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On the gas dynamics of an intense explosion with an expanding contact surface

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The structure of a strong blast wave under the influence of an expanding inner contact surface is studied asymptotically in the Newtonian limit: $\epsilon \equiv (\gamma - 1)/2\gamma \ll 1$, $\epsilon \dot{y}_s^2 \gg a_\infty^2$. The theory treats the interaction of a shock layer and an inner flow region (the entropy wake) and reduces the problem to an ordinary differential equation for the shock radius. The pressure–volume relation of Cheng *et al.* (1961) is recovered and extended to a higher order of ϵ.

It is shown that, depending on the rate of growth of the contact surface, the shock layer may 'reattach' to the surface at large time. In a number of cases, the reattachment is approached in an oscillatory manner which leads to a period of non-uniformity. The associated problem of multiple time scales (treated in sequels to this paper) is identified.

1. Introduction

The unsteady gas dynamics equations admit a self-similar solution for an intense point explosion in a calorically perfect gas (Taylor 1950; Sedov 1959, p. 146–200; Latter 1955; Lin 1954). For the problem considered in this paper, the spatial origin of the blast is allowed to expand as an inner contact surface after the initial explosion (see figure 1). The problem is equivalent to one of an expanding piston, whose motion has been started impulsively, such that the initial energy release is non-zero. The flow field in this case cannot preserve the self-similar form.

The analogy between the plane and cylindrical blast waves and the steady hypersonic flow over blunt nosed flat plate and cylindrical afterbodies is well known (Cheng & Pallone 1956; Lees & Kubota 1957; Chernyi 1959; Hayes & Probstein 1966). It is obvious that the unsteady problem under study corresponds to that of a slightly blunted (planar or axially symmetric) slender body of an arbitrary shape in the hypersonic flow theory. One may note in this connexion the relevance of the present analysis to problems associated with exploding wire experiments and with certain models for sudden expansion of the solar corona (Parker 1963, p. 92–112), as well as the use of blast waves for energy absorption in gases (Diaber, Hertzberg & Wittliff 1966).

In passing, one observes that, with the self-similar solution as an input at small time, numerical integration of the unsteady problem is a straightforward

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matter, since the system is totally hyperbolic. However, the solution to the problem posed has a singularity at the inner boundary which cannot be recovered from standard numerical techniques. The present approach is based on asymptotic solutions for a high shock compression ratio as in the Newtonian flow theory (Cole 1957; Freeman 1960). In the limit of an infinite compression ratio, the

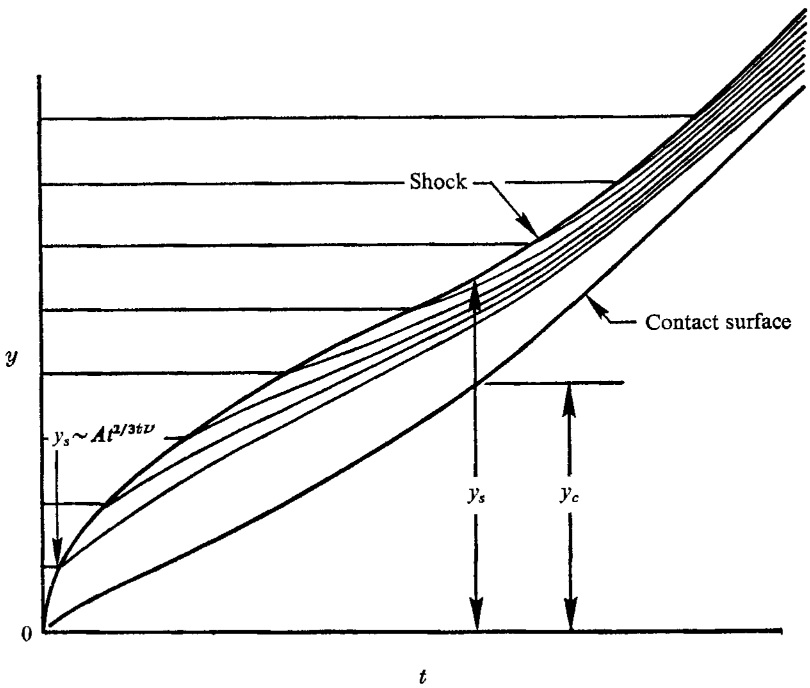

FIGURE 1. A t, y diagram showing movement of a contact surface which affects the field of a blast wave. In the sketch t is the time, y is the distance from the spatial origin of the blast.

mass swept by the shock front of the blast, forms an infinitesimally thin 'shock layer' which travels with the shock, leaving behind it a region of low density and high entropy. The inner region may be referred to as the 'entropy wake' of the blast.† This simplified picture amounts to an application of the Newtonian, or the snow-plow, model to the blast wave problem (illustrated in figure 2). The very same idea has underlaid the earlier work of Chernyi (1959) and Cheng *et al.* (1961) on the equivalent problems of hypersonic flows around slender afterbodies.‡

Existing treatments, however, furnish no knowledge of the inner parts of the density and temperature fields and become arbitrary beyond the leading approxi-

† This is consistent with the term 'entropy wake' used by Hayes & Probstein (1966) for the hypersonic leading-edge problem.

‡ The inner region was referred to previously by Chernyi (1959) and Cheng *et al.* (1961) as the 'entropy layer'.

mation under $\epsilon \equiv (\gamma - 1)/2\gamma \ll 1$, where γ is the specific heat ratio. The present analysis provides solutions to the non-similar field structure in both the low and high density regions, and systematically yields solutions to the second order in ϵ.

The analysis treats, in effect, the interaction of the entropy wake and the shock layer in the presence of an inner expanding contact surface. As a result of the interaction, the zero pressure point of the standard Newtonian theory (Freeman 1956; Cole 1957; Hayes & Probstein 1966) does not appear, nor is it possible for the shock layer to reattach or reimpinge on the piston, except at a large time.

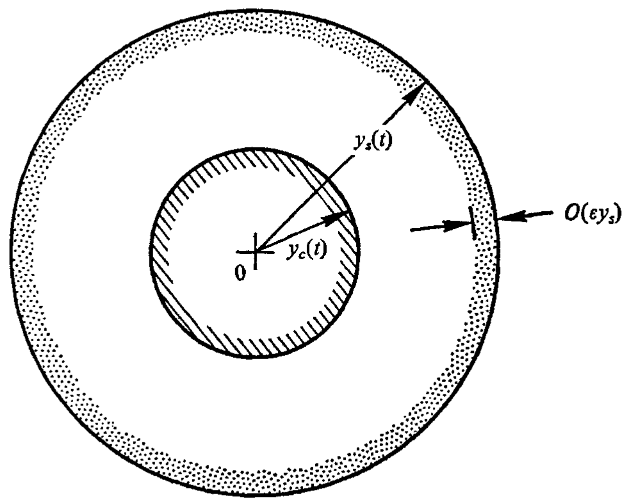

FIGURE 2. The snow-plow model of an intense explosion involving a driving contact surface (Chernyi 1959; Cheng *et al.* 1961).

A condition required for the reattachment (through glancing incidence) at large time is that the contact surface travels much faster than a pure Taylor–Sedov blast wave. Associated with such a reattachment is an oscillatory decay, although examples of non-oscillatory decays cannot be completely ruled out. The oscillation gives rise to a non-uniformity of the present solution at large time and leads to a transition period characterized by multiple time scales. As an example, the case of a contact surface which grows linearly with time in one spatial dimension is studied in some detail.

It may be pointed out that the same type of oscillation has been noted earlier by Cheng *et al.* (1961) in their analyses of hypersonic flow over blunted wedges and cones. Although corresponding results obtained by Chernyi (1959) did not appear to support an oscillatory decay, Schneider (1968) reports recently that an oscillatory decay also exists in Chernyi's solution and can be identified both analytically and numerically.

A part of the study discussed in this paper is based on material from Kirsch (1969), where a number of supporting analyses, computational procedures and details omitted from this paper are reported. Examples of multiple time scale problems are treated in sequels to this paper.

2. Basic equations

Consider an intense, point explosion in a uniform calorically perfect gas, which is followed by a spatially symmetric movement of an inner contact surface as illustration in figure 1. It is stipulated that initially there is a finite amount of energy released by the explosion, and that in the short time duration immediately after the blast, the field is dominated by the constant energy solution. It will be assumed that the fluid motion is particle isentropic and spatially symmetric; that the shock remains strong enough, that the shock Mach number is taken as being infinite; and that the specific heat ratio is close to unity. The shock compression density ratio may therefore be considered as always high like $(\gamma+1)/(\gamma-1)$, or equivalently $\epsilon \equiv (\gamma-1)/2\gamma$ being much smaller than unity.

Let t denote the time, y the distance from the spatial origin, p the pressure, ρ the density and v the velocity. It is possible to define, through the continuity equation under spatial symmetry, a stream function $\psi\ (t, y)$ by

$$\frac{\partial \psi}{\partial y} = \rho y^{\nu}, \quad \frac{\partial \psi}{\partial x} = -\rho v y^{\nu}, \tag{2.1}$$

where $\nu = 0$, 1 and 2 for cases with plane, cylindrical and spherical symmetry, respectively. The partial differential equations governing the problem (with the omission of body forces) in terms of the von Mises variables (t, ψ), are

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} = -y^{\nu}\frac{\partial p}{\partial \psi}, \quad \frac{\partial}{\partial t}\left(\frac{p}{\rho^{\nu}}\right) = 0, \\ &y^{\nu}\frac{\partial y}{\partial \psi} = \frac{1}{\rho}, \quad \frac{\partial y}{\partial t} = v, \end{aligned}\right\} \tag{2.2}$$

where $y(t, \psi)$ determines the particle trajectory. The formulation in (t, ψ) gives a Lagrangian description of the field.

The Rankine–Hugoniot relations provide the outer boundary conditions for the dependent variables. For an initially uniform state, and an infinite shock Mach number, i.e. $\rho_\infty \dot{y}_s^2/\gamma p_\infty \to \infty$, these boundary conditions become

$$p = \frac{2}{\gamma+1}\rho_\infty \dot{y}_s^2, \quad \rho = \frac{\gamma+1}{\gamma-1}\rho_\infty, \quad y = y_s(t) \quad \text{at} \quad \psi = \frac{\rho_\infty}{1+v} y_s^{1+\nu}, \tag{2.3}$$

where $y_s(t)$ is the distance between the shock and the spatial origin. The dot in (2.3) stands for the time derivative and the subscript ∞ refers to the undisturbed condition at infinity. Note that the Rankine–Hugoniot value for v is automatically satisfied. The impermeability condition of the contact surface provides the inner boundary condition

$$v = \dot{y}_c \quad \text{at} \quad \psi = 0, \tag{2.4}$$

where $y_c(t)$ is the distance from the contact surface to the partial origin and is assumed to be differentiable. This is equivalent to the boundary condition $y = y_c(t)$ at $\psi = 0$.

The stipulated dominance of the constant energy solution at the early time requires

$$y_s \sim At^{2/(3+\nu)}, \quad \text{as} \quad t \to 0, \tag{2.5}$$

where A is a constant related to the initial blast strength as well as γ. This, together with the initial energy released E_0, is all one needs for the initial input in the subsequent solution. As may become evident from the results obtained, the requirement (2.5) on the initial shock behaviour imposes a restriction on the contact surface y_c, namely,

$$y_c/t^{2/(3+\nu)} \to 0, \quad \text{as} \quad t \to 0. \tag{2.5a}$$

That is, initially the contact surface shall move more slowly than a pure blast wave front. Equation (2.5) ensures that the total energy in the fluid is bounded; the equivalence of (2.5) and the boundedness requirement on the total energy may be inferred from the earlier works on power law shocks (Lees & Kubota 1957; Chernyi 1959; Mirels 1962). To expedite the subsequent error estimates, it will be stipulated that the approach to the asymptotic limit (2.5) is algebraic:

$$y_s - At^{2/(3+\nu)} \sim Bt^{2/(3+\nu)+\alpha_1} \quad \text{as} \quad t \to 0, \tag{2.5b}$$

where $\alpha_1 > 0$. This implies that $y_c/t^{2/(3+\nu)}$ in (2.5a) is algebraic in t.

3. The region near the shock (shock layer)

3.1. *Proper scales and reduced equations*

For convenience, the stream function ψ will be replaced by $y_*^{1+\nu} \equiv \psi/\rho_\infty(1+\nu)$. The independent variable, y_*, gives the position on the shock at which the particle path associated with ψ leaves the shock, as shown in figure 3. Thus, on the inner

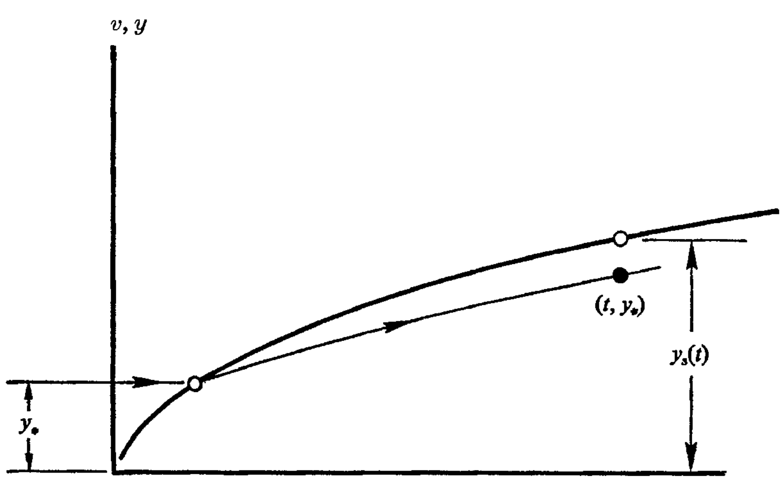

FIGURE 3. Co-ordinate system and definitions of $y_s(t)$ and y_*.

contact surface, one has $y_* = 0$, and at a position right behind the shock, one has $y_* = y_s(t)$.

Let τ and b be the time and length scales, respectively, whose magnitudes will be subsequently chosen. As in classical shock layer theory the characteristic scales in the region near the shock may be inferred from the shock relations, (2.3). Accordingly, one introduces the new dimensionless variables

$$\left.\begin{aligned} t' \equiv t/\tau, \quad Y \equiv y_s/b, \quad Y_* = y_*/b, \quad \hat{y} \equiv (y-y_s)/\epsilon b, \\ \hat{v} \equiv (v-\dot{y}_s)\tau/\epsilon b, \quad \hat{\rho} \equiv \rho\epsilon/\rho_\infty, \quad \hat{p} \equiv p\tau^2/\rho_\infty b^2. \end{aligned}\right\} \tag{3.1}$$

These variables are assumed to be of unit order in the outer region. The prime will be dropped on the t' as a matter of convenience. It is also assumed that the dependent variables have asymptotic expansions in ascending powers of ϵ,

$$\left.\begin{aligned} Y(t) &= Y_0(t)+\epsilon Y_1(t)+O(\epsilon^2), \\ \hat{p}(t,Y_*) &= \hat{p}_0(t,Y_*)+\epsilon\hat{p}_1(t,Y_*)+O(\epsilon^2), \text{ etc.} \end{aligned}\right\} \tag{3.2}$$

The analysis in this outer region is straightforward, and is equivalent to that o Cole (1957), also identifiable with those of Freeman (1956) and Chernyi (1959), specialized to slender bodies.

In terms of the new variables and their expansions, the partial differential equations (2.2) and the shock conditions (2.3) yield, for the leading approximation,

$$\left.\begin{aligned} \ddot{Y}_0 = -\left(\frac{Y_0}{Y_*}\right)^\nu \frac{\partial \hat{p}_0}{\partial Y_*}, \quad \frac{\partial}{\partial t}\left(\frac{\hat{p}_0}{\hat{\rho}_0}\right) = 0, \\ \left(\frac{Y_0}{Y_*}\right)^\nu \frac{\partial \hat{y}_0}{\partial Y_*} = \frac{1}{\hat{\rho}_0}, \quad \hat{v}_0 = \frac{\partial \hat{y}_0}{\partial t}, \end{aligned}\right\} \tag{3.3a}$$

$$\hat{p}_0 = \dot{Y}_0^2, \quad \hat{\rho}_0 = 1, \quad \hat{y}_0 = 0 \quad \text{at} \quad Y_* = Y_0. \tag{3.3b}$$

For the next approximation

$$\left.\begin{aligned} \ddot{Y}_1+\frac{\partial^2\hat{y}_0}{\partial t^2} = \nu\ddot{Y}_0\frac{\hat{y}_0+Y_1}{Y_0}-\left(\frac{Y_0}{Y_*}\right)^\nu\frac{\partial\hat{p}_1}{\partial Y_*} \\ \frac{\partial}{\partial t}\left(\frac{\hat{\rho}_1}{\hat{\rho}_0}-\frac{\hat{p}_1}{\hat{p}_0}+2\ln\hat{p}_0\right) = 0, \quad \hat{v}_1 = \frac{\partial\hat{y}_1}{\partial t}, \\ \nu\frac{Y_1+\hat{y}_0}{Y_0}\frac{1}{\hat{\rho}_0}+\left(\frac{Y_0}{Y_*}\right)^\nu\frac{\partial\hat{y}_1}{\partial Y_*} = -\frac{\hat{\rho}_1}{\hat{\rho}_0^2}, \end{aligned}\right\} \tag{3.4a}$$

$$\hat{p}_1 = 2\dot{Y}_0\dot{Y}_1-\dot{Y}_0^2+Y_1\ddot{Y}_0, \quad \hat{\rho}_1 = -1+3Y_1\ddot{Y}_0/\dot{Y}_0^2, \quad \hat{y}_1 = -1 \quad \text{at} \quad Y_* = Y_0(t). \tag{3.4b}$$

The last terms in each equation of (3.4*b*) arise from transferring the outer boundary condition at $Y_* = Y(t)$ to $Y_* = Y_0(t)$.

The initial blast requirement (2.5) may be written in terms of the adopted variables as

$$\dot{Y}^2 \sim \sigma Y^{-(1+\nu)}[1+O(Y^\alpha)], \quad \sigma \equiv \left(\frac{2\tau}{3+\nu}\right)^2\left(\frac{A}{b}\right)^{3+\nu} \quad \text{as} \quad t\to Y \tag{3.5}$$

with $\alpha > 0$. Only through σ in the above, does the initial energy release enter into the present formulation. For the subsequent analysis, this parameter may be expanded as

$$\sigma = \sigma_0+\epsilon\sigma_1+\ldots, \tag{3.5a}$$

where the coefficients σ_0 and σ_1 will be determined later. One observes from (3.5) for subsequent application that, since $y_s \sim At^{2/3+\nu}$ and $\sigma \propto A^{3+\nu}$, as $t\to 0$,

$$\frac{Y_1}{Y_0} \sim \frac{\dot{Y}_1}{\dot{Y}_0} \sim \frac{\ddot{Y}_1}{\ddot{Y}_0} \sim \frac{1}{3+\nu}\frac{\sigma_1}{\sigma_0}. \tag{3.5b}$$

3.2. *The zeroth- and the first-order outer solutions*

In terms of the zeroth-order shock co-ordinates $Y_0(t)$, (3.3a) and (3.3b) furnish the leading approximation familiar in Cole's (1957) work

$$\left.\begin{aligned}\hat{p}_0 &= \dot{Y}_0^2 + \frac{Y_0\ddot{Y}_0}{1+\nu}[1-(Y_*/Y_0)^{1+\nu}], \quad \hat{\rho}_0 = \hat{p}_0/\dot{Y}_*^2,\\ \hat{y}_0 &= \frac{1}{Y_0^\nu}\int_{Y_0}\frac{\dot{Y}_{0*}^2 Y_*^\nu\,dY_*}{\hat{p}_0(t,Y_*)}, \quad \hat{v}_0 = \frac{\partial \hat{y}_0}{\partial t},\end{aligned}\right\} \tag{3.6}$$

where $\dot{Y}_{0*}$ is a function of Y_* and strictly stands for the value of $\dot{Y}_0(t)$ when $Y_0(t) = Y_*$, i.e. when $t = t_0(Y_*)$ from the inverse of $Y_* = Y_0(t)$. The symbol $\int_{Y_0}$ stands for an integral in Y_* with the lower limit at $Y_* = Y_0$. It results from an inverse transformation of the von Mises variables, and gives the history of the particle associated with each Y_*.

In a similar manner, one obtains from (3.4a) and (3.4b) more lengthy results for the next order, which involves $Y_0(t)$ and $Y_1(t)$,

$$\left.\begin{aligned}\hat{p}_1 &= 2\dot{Y}_0\dot{Y}_1 - \dot{Y}_0^2 + \ddot{Y}_0 Y_1 + \frac{1}{1+\nu}(Y_0\ddot{Y}_1 - \nu\ddot{Y}_0 Y_1)\left[1-\left(\frac{Y_*}{Y_0}\right)^{1+\nu}\right]\\ &\qquad -\frac{1}{Y_0^\nu}\int_{Y_0}\left(\frac{\partial^2}{\partial t^2} - \nu\frac{\ddot{Y}_0}{Y_0}\right)\hat{y}_0 Y_*^\nu\,dY_*,\\ \frac{\hat{\rho}_1}{\hat{\rho}_0} &= \frac{\hat{p}_1}{\hat{p}_0} - 2\ln\hat{p}_0 + 4\ln\dot{Y}_{0*} - 2(\dot{Y}_0\dot{Y}_1 - Y_1\ddot{Y}_0)_*/\dot{Y}_{0*}^2,\\ \hat{y}_1 &= -1 - \hat{y}_0(2Y_1+\hat{y}_0)/2Y_0 - \int_{Y_0}\frac{\hat{\rho}_1}{\hat{p}_0^2}\left(\frac{Y_*}{Y_0}\right)^\nu dY_*, \quad \hat{v}_1 = \frac{\partial\hat{y}_1}{\partial t},\end{aligned}\right\} \tag{3.7}$$

where the subscript * refers to the value at t when $Y_0(t) = Y_*$. Note that the second of (3·6) and (3.7) may be derived from the exact particle isentropy condition.

If the shock layer were adjacent to the contact surface as in Cole's (1957) theory, the impermeable condition there would require $Y(t)+\epsilon y(t,0) = Y_c(t)$, giving $Y_0(t) = Y_c(t)$ and $Y_1(t) = -y_0(t,0)$, $Y_2(t) = -y_1(t,0)$, etc. Equations (3.6) and (3.7) then determine the shock layer structure explicitly in terms of $y_c(t)$, to the order ϵ. However, in the present problem there is a specified singularity, (2.5) or (3.5). Hence, this procedure breaks down because the solution ceases to be valid before the contact surface can be reached, i.e. it becomes invalid as $y_* \to 0$.

To see the breakdown and the need for introducing a distinctly different inner flow region, one may examine the behaviour of the $\hat{y}_0(t,y_*)$ according to (3.6) and (3.7). With $\dot{Y}_{0*}^2 \sim \sigma_0 Y_*^{-(1+\nu)}$ from (3.5), equations (3.6) and (3.7) give, as $Y_* \to 0$

$$\left.\begin{aligned}\hat{y} &\sim \frac{\sigma_0}{Y_0^\nu\hat{p}_0(t,0)}(\ln Y_*) + \epsilon\frac{(1+\nu)\sigma_0}{Y_0^\nu\hat{p}_0(t,0)}\cdot(\ln Y_*)^2 + \dots,\\ \hat{\rho} &\sim \sigma_0^{-1}\hat{p}_0(t,0)\,Y_*^{1+\nu} - \epsilon\,2(1+\nu)\sigma_0^{-1}\hat{p}_0(t,0)\,Y_*^{1+\nu}\,.(\ln Y_*) + \dots,\end{aligned}\right\} \tag{3.8}$$

indicating clearly that the expansions of (3.2) cannot be uniformly valid in the range of Y_* where $\epsilon\ln Y_*$ is of order unity, i.e. where Y_* is exponentially small in ϵ.

4. The inner region (entropy wake)

4.1. *Proper scales and variables*

In seeking a new form of solutions for the inner region where $\epsilon \ln Y_* = O(1)$, one observes that a rescaling of Y_* as Y_*/δ, where δ is a function of ϵ, would not help to resolve the problem of non-uniformity, because the logarithmic singularity remains with the solution in the rescaled variables. It is more expedient to look for a change in the independent variable less restrictive than the affine transform $Y_* \to Y_*/\delta$. There are as many admissible transformations as there are ways to write $\epsilon \ln Y_*$. For example, $(\epsilon \ln Y_*) = \ln(Y_*^{\epsilon}) = (1/n)\ln(Y^{n\epsilon})$, with $n \neq 0, \infty$. The inner variable adopted in the present analysis is

$$\zeta \equiv \frac{Y_*^{2(1+\nu)\epsilon}}{\sigma^{2\epsilon}}, \tag{4.1}$$

which gives solutions in their simplest form. Its choice would have been suggested from studying Sedov's (1959) analytic solution (also see Freeman 1960). The variable is also similar to that employed by Chapkis' (1965) study of power law shocks in the Newtonian limit.

From the definition, it is evident that the zeros of Y_* and ζ coincide and that as soon as the order of Y_* becomes lower than an exponentially small order (in ϵ), ζ is near one. Suffice it to say that Y_* is exponentially small for the inner region. Its absolute scale will not be necessary for the analysis. One may assume that the particle co-ordinates, the gas velocity, and the pressure in this region remain at the same order as in the outer region. The particle isentropy condition (3.8) indicates that the density belongs to the same order as $\dot{Y}_*^{-2}$ i.e. $Y_*^{1+\nu}$, and is therefore exponentially small in ϵ. To accomplish an asymptotic analysis in the inner region, it is expedient to eliminate $Y_*^{1+\nu}$ through a transformation of the density.

The set of new variables fulfilling these requirements is

$$\left.\begin{aligned} t' \equiv t/\tau, \quad \zeta \equiv Y_*^{2(1+\nu)\epsilon}/\sigma^{2\epsilon}, \quad y \equiv y/b, \\ \tilde{v} \equiv v\tau/b, \quad \tilde{p} \equiv p\tau^2/\rho_\infty b^2, \quad \tilde{\rho} \equiv \rho\epsilon/\rho_\infty Y_*^{1+\nu}. \end{aligned}\right\} \tag{4.2}$$

As before, the prime in t' will be dropped.

4.2. *The inner solutions*

In the new variables of (4.2), which all will be taken to be of unit order, the first of (2.2) becomes

$$\frac{\partial \tilde{p}}{\partial \zeta} = -\frac{\sigma}{\tilde{y}^{\nu}}\frac{\partial \tilde{v}}{\partial t}\frac{\zeta^{(1/2\epsilon)}-1}{2(1+\nu)\epsilon}, \tag{4.3}$$

throughout most of the region $0 < \zeta < 1$. The right-hand side of (4.3) is smaller than any power of ϵ. The pressure variation across the region is therefore *exponentially small*, like $\zeta^{1/2\epsilon}$ or $Y_*^{1+\nu}$.

Next, through the definition of $\tilde{\rho}$, the particle isentropy condition, the initia behaviour of the shock, and the fact that the pressure remains uniform, one has

$$\tilde{\rho} = \frac{\hat{\rho}}{Y_*^{1+\nu}} = \frac{\gamma+1}{2\gamma}\left(\frac{\gamma+1}{2}\right)^{1/\gamma}\hat{p}^{1/\gamma}\,\dot{Y}_*^{2/\gamma}\,Y_*^{-(1+\nu)} = \frac{\gamma+1}{2\gamma}\left(\frac{\gamma+1}{2}\right)^{1/\gamma}\frac{\tilde{p}^{1/\gamma}}{\sigma}\frac{1}{\zeta}+0, \tag{4.4}$$

where $\tilde{p}$ is independent of ζ and the relative error is exponentially small, like $(\zeta)^{\alpha/2\epsilon}$. Finally, from the inverse of the von Mises transformation and the inner boundary condition, i.e. $\tilde{y}(t,0)=\tilde{y}_c\equiv y_c/b$,

$$\tilde{y}^{1+\nu}-\tilde{y}^{1+\nu}(t,0)=\frac{1}{2}\int_0^\zeta\frac{d\zeta}{\rho\zeta}=\gamma\left[\frac{2}{\gamma+1}\right]^{1+1/\gamma}\frac{\sigma}{\tilde{p}^{1/\gamma}}\cdot\zeta+0. \tag{4.5}$$

The remainder in (4.5) is exponentially small, belonging to orders $\zeta^{1/2\epsilon}$ and $\zeta^{+1+\alpha/2\epsilon}$. Note that $\tilde{v}=\partial\tilde{y}/\partial t$ follows from (4.5).

Since the relative errors in (4.4) and (4.5) are smaller than any positive integral power of ϵ, the unexpanded form of γ, σ, and $p^{1/\gamma}$ are retained above. These equations can, of course, be rewritten in the regular expansion form by expanding $(\gamma+1)[(\gamma+1)/2]^{1/\gamma}/2\gamma$, σ, and $\tilde{p}^{1/\gamma}$ with respect to ϵ.

Matching the inner region solution to that of the shock layer will provide enough information for the final determination of the shock shape and pressure to $O(\epsilon^2)$. It is essential, however, to verify that a common range of validity for the two asymptotic solutions exist.

5. Reduction to ordinary differential equations—matching of the two solutions

5.1. *The common domain of validity*

Based on equations (4.3)–(4.6), the inner expansion for the particle path, $\tilde{y}(Y_*,t)$, is valid so long as $(1/\epsilon)\zeta^{1/2\epsilon}\ll 1$, i.e. $Y_*^{1+\nu}\ll 1$. According to (3.8), the expansion for the shock layer remains valid for $\epsilon|\ln Y_*|\ll 1$, or equivalently, so long as Y_* is not as small as an exponential order. A common domain of validity for the two expansions may therefore be found in

$$|\epsilon\ln\epsilon|\ll\epsilon|\ln Y_*|\ll 1 \tag{5.1}$$

or, in terms of the inner variable, $\epsilon|\ln\epsilon|\ll|1-\zeta|\ll 1$.

In this range of overlap, one observes that the inner independent variable may be written as

$$\zeta\equiv\frac{Y_*^{2(1+\nu)\epsilon}}{\sigma^{2\epsilon}}=1+[2(1+\nu)\ln Y_*-2\ln\sigma_0]\,\epsilon+O(\epsilon^2\ln^2 Y_*).$$

Consequently, the entropy wake solutions for y from (4.5), in terms of ϵ and Y_*, is

$$\tilde{y}^{1+\nu}\sim Y_c^{1+\nu}(t)+\frac{\sigma_0}{2\tilde{p}_0}+\epsilon\frac{(1+\nu)\,\sigma_0}{\tilde{p}_0}\ln Y_*+\epsilon\frac{\sigma_0}{2\tilde{p}_0}\left[2\ln\left(\frac{\tilde{p}_0}{\sigma_0}\right)+\frac{\sigma_1}{\sigma_0}-\frac{\tilde{p}_1}{\tilde{p}_0}\right]+O(\epsilon^2,\epsilon Y_*^\kappa), \tag{5.2}$$

where Y_*^κ, with $\kappa>0$, represents the remainders contributed by terms of the type $\zeta^{1/2\epsilon}$, $\zeta^{\alpha/2\epsilon}$, i.e. $Y_*^{1+\nu}$ and $Y_*^{\alpha(1+\nu)}$, noted in §4. The corresponding expressions for the outer variables have been omitted to conserve space. Equation (5.2) and the unwritten results for $\tilde{p}$ and $\tilde{\rho}$, corresponds to the 'outer limit of the inner expansion'.

The shock layer expansion for y, from (3.6) and (3.7) may be written for the overlapping region as

$$\left(\frac{y}{b}\right)^{1+\nu} \sim Y_0^{1+\nu}(t)+\epsilon\frac{(1+\nu)\sigma_0}{\hat{p}_0(t,0)}\ln Y_* + \epsilon(1+\nu)\, Y_0^{\nu}(t)\,.\,[Y_1+F\,.\,P\,.\,\hat{y}_0(t,0)]+O(\epsilon^2, \epsilon Y_*^{\kappa}), \qquad (5.3)$$

where

$$-F\,.\,P\,.\,\hat{y}_0(t,0) \equiv -\lim_{Y_*\to 0}\left(\hat{y}_0-\frac{\sigma_0}{Y_0^{\nu}\hat{p}_0}\ln Y_*\right) \qquad (5.3a)$$

is the finite part of the shock layer thickness. Among terms in the remainder of (5.3) are those resulting from $\hat{y}_1$ which are not needed for the level of matching considered in the subsequent section. Equation (5.3) constitutes the 'inner limit of the outer expansion' for the particle path. The corresponding expressions for $\hat{p}$ and $\hat{\rho}$ are omitted for brevity.

5.2. *Matching of the two solutions*

It is readily apparent from (4.3) that matching in pressure dictates that

$$\tilde{p}_0(t) = \hat{p}_0(t,0), \quad \tilde{p}_1(t) = \hat{p}_1(t,0), \ldots,$$

and that matching of the two limiting forms of the density also follows. The overlap expansions for the particle trajectories, i.e. equations (5.2) and (5.3), may also be matched (which also ensures matching in the velocity) to the first order. This yields the two equations

$$\left.\begin{aligned} \tilde{p}_0(t)\,[Y_0^{1+\nu}(t)-Y_c^{1+\nu}(t)] &= \tfrac{1}{2}\sigma_0,\\ \frac{\tilde{p}_1(t)}{\tilde{p}_0(t)}+(1+\nu)\,\tilde{p}_0(t)\frac{2}{\sigma_0}\,Y_0^{\nu}[Y_1+F\,.\,P\,.\,\hat{y}_0(t,0)] &= \frac{\sigma_1}{\sigma_0}+2\ln\left(\frac{\tilde{p}_0(t)}{\sigma_0}\right).\end{aligned}\right\} \qquad (5.4)$$

The first of (5.4) is recognized as the pressure–volume relation obtained by Cheng *et al.* (1961). In the second equation, it is seen that there is a linear relationship between the surface pressure correction, $\tilde{p}_1(t)$, the shock displacement, $Y_1(t)$, and an effective shock layer thickness, $F\,.\,P\,.\,\hat{y}_0(t,0)$. With $\tilde{p}_0(t)$ given by the Busemann (1933, pp. 244–279) pressure formula and $\tilde{p}_1(t)$, through (3.7) and (3.7a), by

$$\tilde{p}_1(t) = 2\dot{Y}_0\dot{Y}_1+\frac{Y_0\ddot{Y}_1+\ddot{Y}_0Y_1}{1+\nu}+\frac{1}{(1+\nu)Y_0^{\nu}}\cdot\left(\frac{\partial^2}{\partial t^2}-\nu\frac{\ddot{Y}_0}{Y_0}\right)\frac{1}{Y_0^{\nu}}\int_{Y_0}^{0}\frac{\dot{Y}_{0*}^2\,Y_*^{1+2\nu}\,dY_*}{\hat{p}_0(t,Y_*)}. \qquad (5.5)$$

Equation (5.4) yields a non-linear second-order ordinary differential equation for Y_0 and a linear second-order equation for Y_1

$$\left.\begin{aligned} \frac{1}{1+\nu}Y_0\ddot{Y}_1+2\dot{Y}_0\dot{Y}_1+\left\{\frac{1}{1+\nu}\ddot{Y}_0+(1+\nu)\,[\tilde{p}_0]^2\frac{2}{\sigma_0}\,Y_0^{\nu}\right\}.\,Y_1 &= W(t),\\ \left[\frac{1}{1+\nu}Y_0\ddot{Y}_0+\dot{Y}_0^2\right][Y_0^{1+\nu}-Y_c^{1+\nu}] &= \frac{\sigma_0}{2},\end{aligned}\right\} \qquad (5.6)$$

where $W(t)$, omitted to conserve space, is determined explicitly by $Y_0(t)$, σ_0 and σ_1.

The constants σ, hence σ_0 and σ_1, may be related to the initial blast strength and γ through.

$$\sigma = \frac{2(\gamma^2-1)}{k_\nu I_b}\frac{E_0\tau^2}{\rho_\infty b^{3+\nu}}, \tag{5.7}$$

where $k_\nu = 1$, 2π and 4π for $\nu = 0$, 1, and 2, respectively, and

$$I_b \equiv \int_0^1 (\bar{P}+\bar{R}\bar{V}^2)\,\eta^\nu\,d\eta = \frac{1}{2(1+\nu)}[1+\epsilon(3-2\ln 2)+O(\epsilon^2)], \tag{5.7a}$$

with $\bar{P}$, $\bar{R}$, $\bar{V}$ and η being the ratios of the pressure, density, velocity, and y to their shock values in the case of a pure blast wave. One may now choose the reference scales τ and b so that $\frac{1}{2}\sigma_0$ is unity.† With this choice, (5.7) and (5.7a) give

$$\sigma_0 = 2, \quad \sigma_1 = 4\ln 2. \tag{5.7b}$$

The fact that σ_0 and σ_1 in (5.4) and (5.6) are of unit order insures that the effect of the initial blast does not appear to be small when $t = O(1)$.

It should be noted that the equations in (5.4) may be combined to yield

$$\left.\begin{aligned} 2\left[\frac{\hat{p}(t,0)}{\sigma}\right]^{1/\gamma}[Y_e^{1+\nu}(t)-Y_c^{1+\nu}(t)] &= 1+O(\epsilon^2),\\ Y_e \equiv Y(t)+\epsilon F.P.\hat{y}_0(t,0). \end{aligned}\right\} \tag{5.8}$$

This is similar to the pressure–volume relations stipulated in earlier studies of Guiraud (1965), Cheng *et al.* (1961) and Mirels (1962). However, the constant on the right-hand side of the present relation is completely determined to the order ϵ. Moreover, with the appearance of the finite part of the shock layer integral, the base of the shock layer is now unambiguously identified with the function $Y_e(t)$.

5.3. *The solutions at small and large time*

Under condition (2.5a), the small time limit of (5.6) becomes

$$Y_0 d^2Y_0^{2+\nu}/dt^2 = (1+\nu)(2+\nu), \tag{5.9}$$

which is reducible to a first-order differential equation for Y_0. There is a single integral curve among the one-parameter family, which gives the pure blast-wave solution $Y_0 \propto t^{2/(3+\nu)}$. A particular integral to the second of (5.6) for Y_1 in the limit $t\to 0$ is $Y_1 \sim \sigma_1 Y_0/2(3+\nu)$, which is the only integral curve that fulfills the required pure blast behaviour (3.5b). The integral curves at large Y_0 and Y_1 all tend to the blast wave solution as $t\to 0$. Hence, the pure blast solution is stable for an increasing t. Thus, with the help of a series solution for small time, numerical solutions to (5.6) can be carried forward in t. The singular solutions for Y_0 and Y_1 at small t yield

$$\left.\begin{aligned} Y &\sim 2^{1/(3+\nu)}\left[1+\epsilon\frac{2\ln 2}{3+\nu}\right]\left(\frac{3+\nu}{2}t\right)^{2/(3+\nu)},\\ \tilde{p} &\sim 2^{-(1+\nu)/(3+\nu)}\left[1-\epsilon\frac{2+\nu}{3+\nu}4\ln 2\right]\left(\frac{3+\nu}{2}t\right)^{-2(1+\nu)/(3+\nu)} \end{aligned}\right\} \tag{5.10}$$

† The required condition is $b^{3+\nu}/\tau^2 = 2(1+\nu)\epsilon E_0/k_\nu\rho_\infty$.

For $\gamma = \frac{7}{5}$, the difference between the exact similar solutions and the corresponding expressions in (5.10) is found to be 2–4 % in shock shape, and 3–5 % in pressure. The improvements accrued by including the first-order correction will be further demonstrated in a subsequent section.

Consider now the behaviour at long time under the assumption that

$$Y_c \sim t^{\omega}, \quad \text{as} \quad t \to \infty, \tag{5.11}$$

where ω is a constant exponent. The first of (5.6) admits a power law solution for $t \to \infty$, depending on ω,

$$Y_0 \sim t^{2/(3+\nu)}, \quad \text{if } \omega \leqslant \frac{2}{3+\nu}; \; Y_0 \sim t^{\omega}, \quad \text{if } \omega > \frac{2}{3+\nu}. \tag{5.12}$$

In the case of $\omega \leqslant 2/(3+\nu)$, the differential equations governing Y_0 and Y_1 can be reduced for large t to the same form as for small t, and the solution represented by the first of (5.12) is readily seen to be a stable one, inasmuch as $t^{\omega}/Y \to 0$. In the case $\omega > 2/(3+\nu)$, the solution indicated by the second of (5.12) is also stable, since subsequent study in §6 will reveal that the solution indicated is the asymptotic limit for a two-parameter family of integral curves.

The change of behaviour at $\omega = 2/(3+\nu)$ signifies that, at large t, reattachment of the shock layer to the contact surface will occur for $\omega > 2/(3+\nu)$. Equation (5.12) in fact bears out the form of asymptotic solution proposed by Freeman *et al.* (1964) for a power law afterbody. The second of (5.6), or (5.4), for $\omega > 2/(3+\nu)$ leads to the large-time behaviour for the next approximation of Y,

$$Y_1 \sim F \,.\, P \,.\, \hat{y}_0(t, 0) \propto t^{\omega}. \tag{5.13}$$

This result is readily arrived at by considering the composite form (5.8), which under $Y_c/t^{2/(3+\nu)} \to \infty$, yields $[Y(t) - Y_c(t)] \sim \epsilon F \,.\, P \,.\, \tilde{y}_0(t, 0)$. It agrees with the Newtonian results (Cole 1957) in the absence of an initial blast.

6. Oscillatory approach to the reattachment at large t

The equation (5.6) admits solutions for the shock shape which exhibit an oscillatory decay with a distinct frequency and damping rate. The following discuss the periods of non-uniformity of the expansions associated with the oscillation and will note the character of multiple time scales for the proper description of these periods.

Consider the case which is equivalent to a blunted wedge in hypersonic flow, i.e. $Y_c = t$, $\nu = 0$. The differential equations (5.6) and corresponding equations in the next order admit solutions for large t, giving

$$\begin{aligned}\hat{p}(t) \sim 1 - \frac{A_0}{t^{\frac{3}{4}}} \cos\{2\sqrt{(t)} + \phi_0\} + \ldots - \epsilon\Big[&(B_0 t^{\frac{3}{4}} + B_1 t^{\frac{1}{4}}) \cos\{2\sqrt{(t)} + \phi_0\} + 1 + \ldots \\ &+ \frac{A_1}{t^{\frac{3}{4}}} \cos\{2\sqrt{(t)} + \phi_1\} + \ldots\Big] - \epsilon^2[C_0 t^{\frac{9}{4}} \cos\{2\sqrt{(t)} + \phi_0\} + \ldots] + \ldots,\end{aligned} \tag{6.1}$$

where A_0, ϕ_0, A_1 and ϕ_1 are constants of integration and the constants B_0, B_1 and C_0 are determined by A_0. The form of (6.1) has been confirmed by numerical integration of (5.6) with $A_0 \simeq 0{\cdot}71$, $\phi_0 \simeq 79°$ (see discussion in § 7).

Obviously, the solution fails at large t. Two periods of non-uniformity may be identified

$$\text{I} \;\; t = O(\epsilon^{-\frac{2}{3}}); \quad \text{II} \;\; t = O(\epsilon^{-1}). \tag{6.2}$$

The first range of (6.2) follows directly from (6.1); it is not difficult to show that this is the period of non-uniformity for the m term expansion ($m \geqslant 2$) of $\hat{y}$ and $\hat{p}$ in the case $Y_c = t$, $\nu = 0$. The second range of (6.2) could have been arrived at through equating the thicknesses of the entropy-wake and the shock layer, since implicit in the foregoing analysis is the assumption that the former is much thicker than the latter. The need for the second range, II, is verified by an analysis of range I (to be delineated in sequels to this paper). The same analysis also shows that range I is less important, since it does not contribute to the leading order of $\tilde{p}-1$. It should be apparent, in the meantime, that the oscillation revealed in (6.1) imposes a time scale which is much smaller than either of the two in (6.2). Thus, the problem in either period is described by two characteristic times as in the multiple time scale method (see Cole 1968; Van Dyke 1964).

For a more general contact surface, for which $Y_c \sim t^{\omega}$ with a remainder being algebraic in t, the equation for $Y_0(t)$, (5.6), admits an oscillatory approach for $2/(3+\nu) < \omega < 4$ (Kirsch 1969)

$$Y_0 \sim Y_c + g(t;\omega,\nu) + \frac{A_0}{t^{\lambda_2}} \cos\left[f(\omega,\nu)\, t^{\lambda_1} + \phi_0\right];$$

$$\lambda_1 \equiv \frac{3+\nu}{2}\omega - 1, \quad \lambda_2 \equiv \frac{7+5\nu}{4} - 1, \quad f(\omega,\nu) \equiv \frac{2\omega[(2+\nu)\omega - 1]}{(3+\nu)\omega - 2}, \tag{6.3}$$

where g is an algebraic function of t, ω and ν; A_0 and ϕ_0 are constants of integration. However, examples with a non-oscillatory approach (implying $A_0 = 0$) may still be constructed from solutions to the *inverse* problem. For example, the contact surface supporting a non-oscillatory shock, $Y_0 = [(3+\nu)\,t/2]^{2/(3+\nu)} + t^{\omega}$ can be computed from the first of (5.6) and is non-oscillatory. For $A_0 \neq 0$, the periods of non-uniformity corresponding to ranges I and II of (6.2) are respectively,

$$\text{I} \;\; t = O(\epsilon^{-\frac{1}{3}\lambda_1}); \quad \text{II} \;\; t = O(\epsilon^{-\frac{1}{2}\lambda_1}).$$

The time scale characterizing the oscillation in (6.3) then appear to belong to an order $\epsilon^{\frac{1}{3}}$ higher than range I, and $\epsilon^{\frac{1}{2}}$ higher than range II. It should be noted that the anomaly of the theory presented in § 5 appears mainly in the higher approximations. In the case of $Y_c = t$, the pressure corrections in the important range II are of the order $\epsilon^{\frac{3}{4}}$ for $\nu = 0$, and of $\epsilon^{\frac{1}{2}}$ for $\nu = 1$.

7. A numerical example: $Y_c = t$, $\nu = 0$

As an example, solutions to (5.6) will be studied for the case $Y_c = t$ and $\nu = 0$, corresponding to a slightly blunted thin wedge in a hypersonic flow. Comparison with existing numerical characteristic solutions for the blunted wedge problem may indicate the degree of improvement over the earlier work that can be attained through the present analysis.

With the help of series solution for small time, solutions to (5.6) are obtained through forward integration in t. The integration (executed on a Honeywell 800) employs a Runge–Kutta procedure over five decades of the reduced time, t.

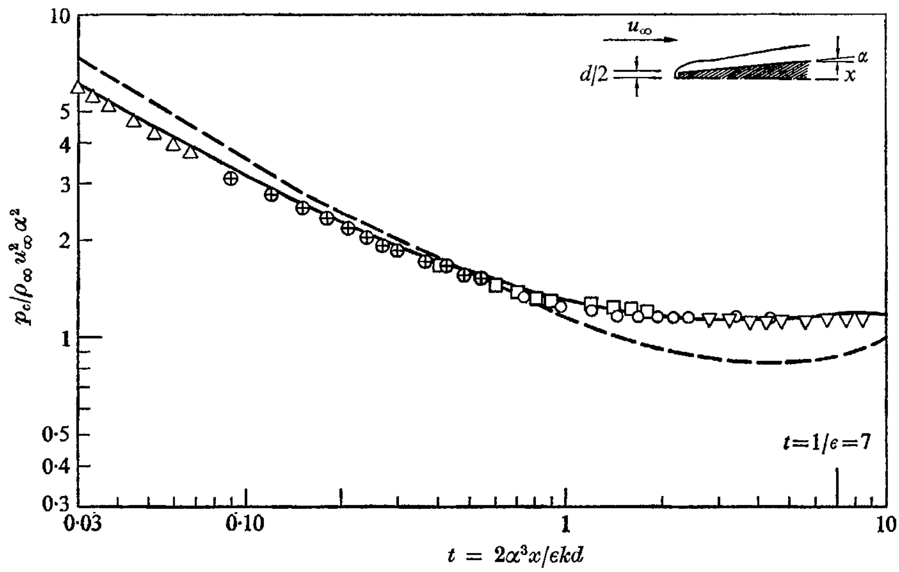

FIGURE 4. Comparison of the pressure on the contact surface $y_c = t$ based on the present analysis with pressure on blunted wedges in hypersonic flows. Solution by characteristics method. Half wedge angle α when $M_\infty = 10^4$, $\gamma = 1{\cdot}40$, $x/d > 2$ (Cleary & Axelson 1964): $\triangle$, 5°; $\oplus$, 10°; $\square$, 15°; $\circ$, 20°; ∇, 25°. — — —, zeroth-order solution, $\tilde{p}_0$; ——, present solution, $\tilde{p}_0 + \epsilon\tilde{p}_1$.

A uniform time step, Δt, is chosen for each decade, with 100 points per decade. Some care is required in evaluating the finite-part and other integrals in the function $W(t)$ of (5.6). The solutions for Y_0 and $\tilde{p}_0$ are indistinguishable from the earlier results of Cheng *et al.* (1961) which was computed with a different procedure (and step sizes). Over the range of $t = 10$ to $t = 100$, the large time solutions for $\tilde{p}_0$ and Y_0 with $A_0 \simeq 0{\cdot}71$ and $\phi_0 \simeq 79°$ are confirmed to within 2 %.

For a meaningful comparison with the hypersonic wedge flow, the scales τ and b are related in accordance with the equivalence principle to the flow speed u_∞, the nose diameter d, nose drag coefficient k, and the half wedge angle α as: $\tau = \epsilon kd/2\alpha^3 u_\infty$ and $b = \epsilon kd/2\alpha^2$. Thus,

$$t = \frac{2\alpha^3 x}{\epsilon kd}, \quad Y = \frac{2\alpha^2 Y_s}{\epsilon kd}, \quad \tilde{p} = \frac{p}{\rho_\infty u_\infty^2 \alpha^2}, \tag{7.1}$$

where x is the distance from the nose. In figure 4, the surface pressure on blunted wedges obtained by Cleary & Axelson (1964) from the characteristics method for $\gamma = \frac{7}{5}$ ($\epsilon = \frac{1}{7}$) and an infinite Mach number are correlated through (7.1) as $\tilde{p}$ *versus* t (open dots). To satisfy the small disturbance assumption required by the equivalence principle, data for $\alpha > 20°$ and $x/d < 2$ are excluded. The result $\tilde{p}_0 + \epsilon\tilde{p}_1$ based on the present work is plotted as a full line. According to §5, the transition period where the first-order corrections become invalid, occurs at t of the order $1/\epsilon$ (which is 7 for $\gamma = \frac{7}{5}$). The curve is therefore not reproduced beyond $t = 10$. For comparison, the 'zeroth-order approximation', p_0, originally given by Cheng *et al.* (1961), is also included (dash curve).

The improvement of the present work over the zeroth-order solution in the range of t shown is obvious; reasonably good agreement of the present result with the correlated characteristic solution is also apparent. The small difference from the characteristic solution, even at $t = 10$, is not unexpected, since the relative error in the pressure is at most $O(\epsilon^{\frac{3}{4}})$ at $t = O(1/\epsilon)$ as noted earlier. One may also note that the large pressure undershoot found in the zeroth-order solution almost disappears in the present result and that the limit corresponding to a sharp wedge has been quite closely approached even before the transition period. There is also reasonable agreement between Cleary's (1965) numerical solution and the present analysis in the shock shape (see Kirsch 1969).

While the precise error of the blast-wave analogy (Lees & Kubota 1957; Cheng & Pallone 1956) remains unsettled (see Hayes & Probstein 1966), it must be pointed out that the correlated data from characteristics solutions in figure 6 of that paper are themselves a measure of the degree of validity of the equivalence principle, as much as the pressure is concerned.

8. Concluding remarks

A gas dynamics theory has been presented for an intense explosion which involves an expanding inner contact surface under the conditions: $\epsilon \dot{y}_s^2 \gg a_\infty^2$, $\epsilon \ll 1$. The interaction between the entropy wake of the initial blast and the shock layer is analyzed and the structure of the two flow regions is determined. In the process, the pressure-volume relation obtained from an earlier model (Cheng *et al.* 1961) is recovered and is extended to a higher order in ϵ.

The reduced differential equations admit a reattachment solution accompanied by an oscillatory decay for a class of contact surface motion. This oscillation, as confirmed in the numerical examples discussed in the text, leads to non-uniformity at large t. Two transition periods are identified, each is described by two local time scales. Solution of this 'two-time problem' may furnish a theoretical basis for modelling the reattachment process in the shock layer approximation.

Comparison of the present results for the case $Y_c = t$, $\nu = 0$ with the numerical characteristics solution of the equivalent problem of hypersonic wedge flow has been made. Substantial improvement over the 'zeroth-order approximation' (Cheng *et al.* 1961) in the range of the scaled time $0 < t < O(1/\epsilon)$ is demonstrated.

Finally, a comment should be made on the model of a blast wave involving an asymmetrical movement of the inner contact surface. The highly stratified density field described by (4.4) and (4.5) suggests that flow symmetry will be confined principally to a restricted, inner region of the entropy wake. This inner wake region is bounded by the smallest constant Y_* surface generated from the symmetric solutions which encloses the asymmetric contact surface. Such a model corresponds to the 'hypersonic area rule' proposed by Ladyzheniski

(1961) in the eqivalent blunt-nosed slender body problem, but represents a form stronger than the latter.

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